ISRAEL JOURNAL OF MATHEMATICS **141** (2004), 355–368

THE MONODROMY GROUP OF A FUNCTION ON A GENERAL CURVE

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ABSTRACT

Let C_g be a general curve of genus $g \geq 4$. Guralnick and others proved that the monodromy group of a cover $C_g \to \mathbb{P}^1$ of degree n is either S_n or A_n. We show that A_n occurs for $n \geq 2g+1$. The corresponding result for S_n is classical.

1. Introduction

Let C_g be a general curve of genus $g \geq 2$ (over $\mathbb{C}$). Then C_g has a cover to $\mathbb{P}^1$ of degree n if and only if $2(n-1) \geq g$. This is a classical fact of algebraic geometry. (It is part of Brill–Noether theory, which more generally considers maps of a curve to $\mathbb{P}^m$, see [ACGH], Ch. V, in particular Thm. 1.1 or [HM], Ch. 5.) If C_g has a cover to $\mathbb{P}^1$ of degree n, then there is such a cover that is simple, i.e., has monodromy group S_n and all inertia groups are generated by transpositions (cf. Remark 3.5 below) The question arises whether C_g admits other types of covers to $\mathbb{P}^1$

* Partially supported by NSA grant MDA-9049810020.
** Partially supported by NSF grant DMS-0200225.
Received March 10, 2003 and in revised form September 22, 2003

If there is a cover $C_g \to \mathbb{P}^1$ branched at r points of $\mathbb{P}^1$ and $g \geq 2$ then $r \geq 3g$ (see Remark 2.2 below). Zariski [Za] used this to show that if $g > 6$ then there is no such cover with solvable monodromy group. He made a conjecture on the existence of such covers for $g \leq 6$, but there is a counterexample to that, see Fried [Fr2], Fried/Guralnick [FrGu].

The condition $r \geq 3g$ was further used by Guralnick to restrict the possibilities for the monodromy group G of a cover $C_g \to \mathbb{P}^1$ of degree n. Assume the cover does not factor non-trivially, i.e., G is a primitive subgroup of S_n. (Knowledge of this case is sufficient to know all types of covers $C_g \to \mathbb{P}^1$; this was already observed by Zariski [Za], see [GM].) If further $g > 3$, then $G = S_n$ or $G = A_n$ For $g = 3$ there are 3 additional cases, with $n = 7, 8, 16$ and $G = GL_3(2), AGL_3(2), AGL_4(2)$, respectively. This was proved by Guralnick and Magaard [GM] and Guralnick and Shareshian [GS] using the classification of finite simple groups. There is also a corresponding result for $g = 2$, but it is less definitive.

As noted in [GM], it was not known whether the case $G = A_n$ actually occurs. This is answered in the affirmative in this paper. More precisely, we prove the following: Let $g \geq 3$ and $n \geq 2$. Then the general curve of genus g admits a cover to $\mathbb{P}^1$ of degree n with monodromy group A_n such that all inertia groups are generated by double transpositions if and only if $n \geq 2g + 1$. The same statement holds when we replace double transpositions by 3-cycles (see Theorem 3.3). We refine the latter result in Theorem 4.1 by showing that both of the two types of 3-cycle covers occur for the general curve. (See Fried [Fr1] and Serre [Se1], [Se2] for this type distinction.) We also study the exceptional cases in genus 3.

A preliminary version of this paper has been circulated since October 2001. It was brought to our attention that in a recent preprint S. Schröer [Schr] proves a weaker version of our result on 3-cycles (which, however, also holds in positive characteristic): The locus in $\mathcal{M}_g$ of curves admitting a cover to $\mathbb{P}^1$ with only triple ramification points has dimension $\geq \max(2g - 3, g)$.

ACKNOWLEDGEMENT: The authors are grateful to Bob Guralnick for raising the question of moduli dimension for alternating groups. We further gratefully acknowledge the kind support of Gerhard Frey without whose expert advice this paper may have never been finished.

2. Moduli dimension of a tuple in S_n

Let $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ be the Riemann sphere. Let $\mathcal{U}^{(r)}$ be the open subvariety of $(\mathbb{P}^1)^r$

consisting of all $(p_1, \ldots, p_r)$ with $p_i \neq p_j$ for $i \neq j$. Consider a cover $f\colon X \to \mathbb{P}^1$ of degree n, with branch points $p_1, \ldots, p_r \in \mathbb{P}^1$. Pick $p \in \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$, and choose loops γ_i around p_i such that $\gamma_1, \ldots, \gamma_r$ is a standard generating system of the fundamental group $\Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}, p)$ (see [V], Thm. 4.27); in particular, we have $\gamma_1 \cdots \gamma_r = 1$. Such a system $\gamma_1, \ldots, \gamma_r$ is called a homotopy basis of $\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$. The group Γ acts on the fiber $f^{-1}(p)$ by path lifting, inducing a transitive subgroup G of the symmetric group S_n (determined by f up to conjugacy in S_n). It is called the **monodromy group** of f. The images of $\gamma_1, \ldots, \gamma_r$ in S_n form a tuple of permutations called a tuple of **branch cycles** of f.

Let $\sigma_1, \ldots, \sigma_r$ be elements $\neq 1$ of the symmetric group S_n with $\sigma_1 \cdots \sigma_r = 1$, generating a transitive subgroup. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$. We call such a tuple **admissible**. We say a cover $f\colon X \to \mathbb{P}^1$ of degree n is of type σ if it has σ as tuple of branch cycles relative to some homotopy basis of $\mathbb{P}^1$ minus the branch points of f. The genus g of X depends only on σ (by the Riemann–Hurwitz formula); we write $g = g_\sigma$.

Let $\mathcal{H}_\sigma$ be the set of pairs $([f], (p_1, \ldots, p_r))$, where $[f]$ is an equivalence class of covers of type σ, and $p_1, \ldots, p_r$ is an ordering of the branch points of f. We use the usual notion of equivalence of covers, see [V], p. 67. Let $\Psi_\sigma\colon \mathcal{H}_\sigma \to \mathcal{U}^{(r)}$ be the map forgetting $[f]$. The **Hurwitz space** $\mathcal{H}_\sigma$ carries a natural structure of quasiprojective variety such that Ψ_σ is an algebraic morphism, and an unramified covering in the complex topology (see [FrV], [V], [We]). We also have the morphism

$$\Phi_\sigma\colon \mathcal{H}_\sigma \to \mathcal{M}_g$$

mapping $([f], (p_1, \ldots, p_r))$ to the class of X in the moduli space $\mathcal{M}_g$ (where $g = g_\sigma$). Each irreducible component of $\mathcal{H}_\sigma$ has the same image in $\mathcal{M}_g$ (since the action of S_r permuting $p_1, \ldots, p_r$ induces a transitive action on the components of $\mathcal{H}_\sigma$). Hence this image, i.e., the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ, is irreducible.

Definition 2.1: (a) The moduli dimension of σ, denoted by **mod-dim**(σ), is the dimension of the image of Φ_σ; i.e., the dimension of the locus of genus g curves admitting a cover to $\mathbb{P}^1$ of type σ. We say σ has **full moduli dimension** if **mod-dim**$(\sigma) = \dim \mathcal{M}_g$.

(b) We say σ has **infinite moduli degree** if the following holds: If $f\colon X \to \mathbb{P}^1$ is a cover of type σ with general branch points then X has infinitely many covers to $\mathbb{P}^1$ of (the same) type σ such that the corresponding subfields of the function field of X are all different. (This terminology is further discussed at the end of this section.)

A curve is called a **general curve of genus** g if it corresponds to a point of $\mathcal{M}_g$ that does not lie in any proper closed subvariety of $\mathcal{M}_g$ defined over $\bar{\mathbb{Q}}$ (the algebraic closure of the rationals). Clearly, an admissible tuple σ has full moduli dimension if and only if each general curve of genus g_σ admits a cover to $\mathbb{P}^1$ of type σ.

Part (a) of the following Remark is the necessary condition for full moduli dimension used by Guralnick, Fried and Zariski. We indicate the proof at the end of this section.

Remark 2.2: Let σ be an admissible tuple of length r in S_n, and $g := g_\sigma$.

(a) Suppose σ has full moduli dimension. Then $r-3 \geq \dim \mathcal{M}_g$, thus if $g \geq 2$ then $r \geq 3g$.

(b) If $r-3 > \dim \mathcal{M}_g$ then σ has infinite moduli degree.

Here is a simple but crucial lemma that allows us to make use of the hypothesis of infinite moduli degree.

LEMMA 2.3: *Suppose f_i: $X \to \mathbb{P}^1$ is an infinite collection of covers such that the corresponding subfields of the function field of X are all different. Let S be the set of $(x,y) \in X \times X$ with $f_i(x) = f_i(y)$ for some i. Then S is Zariski dense in $X \times X$.*

Proof: Let S_i be the curve on $X \times X$ consisting of all (x,y) with $f_i(x) = f_i(y)$. The set S is the union of all S_i. If S is not Zariski dense in $X \times X$ then it must be the union of finitely many S_i; then the curves S_i cannot be all distinct. But if $S_i = S_j$ then the subfields of $\mathbb{C}(X)$ corresponding to f_i and f_j coincide. This contradicts the hypothesis. ∎

Here is our sufficient condition for full moduli dimension.

LEMMA 2.4: *Let $n \geq 3$. Given an admissible tuple $\sigma = (\sigma_1, \ldots, \sigma_r)$ in S_n with $g_\sigma > 0$, define $\tilde{\sigma} = (\sigma_1, \ldots, \sigma_{r+2})$, where either*

$$\sigma_{r+1} = \sigma_{r+2} = (1,2)(n,n+1)$$

is a double transposition or

$$\sigma_{r+1} = \sigma_{r+2}^{-1} = (n-1,n,n+1)$$

is a 3-cycle. Then $\tilde{\sigma}$ is an admissible tuple in S_{n+1} with $g_{\tilde{\sigma}} = g_\sigma + 1$. If σ has infinite moduli degree then

$$\textbf{mod-dim}(\tilde{\sigma}) \geq \textbf{mod-dim}(\sigma) + \begin{cases} 3 & \text{if } g_\sigma > 1, \\ 2 & \text{if } g_\sigma = 1. \end{cases}$$

Proof: Let $g := g_\sigma$. Then $g_{\tilde\sigma} = g + 1$ by Riemann–Hurwitz. Let $\Phi := \Phi_{\tilde\sigma}$ and $\mathcal{H} := \mathcal{H}_{\tilde\sigma}$. The map Φ extends to $\bar\Phi\colon \bar{\mathcal{H}} \to \bar{\mathcal{M}}_{g+1}$, where $\bar{\mathcal{M}}_{g+1}$ is the stable compactification of $\mathcal{M}_{g+1}$, and $\bar{\mathcal{H}}$ is $\mathcal{H}$ plus that piece $\partial\mathcal{H}$ of the boundary where the last two branch points come together (see [We]); thus $\bar{\mathcal{H}}$ covers the set of $(p_1, \ldots, p_{r+2})$ in $(\mathbb{P}^1)^{r+2}$ with $p_i \neq p_j$ for $i \neq j$ unless $\{i, j\} = \{r+1, r+2\}$, and $\partial\mathcal{H}$ is the inverse image of the subset defined by the condition $p_{r+1} = p_{r+2}$.

If we coalesce the last two entries of $\tilde\sigma$ we obtain σ, which has orbits of length n and 1 on $\{1, \ldots, n+1\}$. For a cover $X_{g+1} \to \mathbb{P}^1$ of type $\tilde\sigma$, this means the following: When coalescing the last two branch points, X_{g+1} degenerates into a nodal curve $\bar X$ with two components linked at one point P. One component is a non-singular curve covering $\mathbb{P}^1$ of degree 1. The other component $\bar X_g$ is a singular curve whose only singularity is a node N. Its normalization X_g covers $\mathbb{P}^1$ of type σ. If $\sigma_{r+1} = (1,2)(n, n+1)$ then N corresponds to the cycle $(1,2)$ and P to the cycle $(n, n+1)$. If $\sigma_{r+1} = (n-1, n, n+1)$ then $N = P$.

The nodal curve $\bar X$ is stably equivalent to the stable curve $\bar X_g$, and the latter constitutes the image in $\bar{\mathcal{M}}_{g+1}$ of the element of $\partial\mathcal{H}$ corresponding to $\bar X \to \mathbb{P}^1$ (see [HM], Th. 3.160). Thus the image of $\partial\mathcal{H}$ in $\bar{\mathcal{M}}_{g+1}$ lies in the boundary component consisting of irreducible curves with one node whose normalization has genus g. We can identify this boundary component with $\mathcal{M}_{g,2}$ (= moduli space of genus g curves with two unordered marked points). The two marked points correspond to the node. Thus we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_\sigma & \xrightarrow{\Phi_\sigma} & \mathcal{M}_g \\
\uparrow & & \uparrow \\
\partial\mathcal{H} & \longrightarrow & \mathcal{M}_{g,2} \\
\downarrow & & \downarrow \\
\bar{\mathcal{H}} & \xrightarrow{\bar\Phi} & \bar{\mathcal{M}}_{g+1}
\end{array}$$

where the vertical arrows on the lower level are inclusion. The map $\mathcal{M}_{g,2} \to \mathcal{M}_g$ is the natural projection (forgetting the marked points), and the map $\partial\mathcal{H} \to \mathcal{H}_\sigma$ sends the point corresponding to the cover $\bar X \to \mathbb{P}^1$ to that corresponding to the cover $X_g \to \mathbb{P}^1$ of type σ (see the previous paragraph).

The image of $\bar{\mathcal{H}}$ in $\bar{\mathcal{M}}_{g+1}$ is irreducible (see the remarks before Definition 2.1). Its intersection with the boundary of $\bar{\mathcal{M}}_{g+1}$ is a closed proper subvariety, hence has codimension at least 1. This subvariety contains the image of $\partial\mathcal{H}$, which we denote by $\mathrm{Im}(\partial\mathcal{H})$. Thus $\textbf{mod-dim}(\tilde\sigma) \geq 1 + \dim \mathrm{Im}(\partial\mathcal{H})$.

The fiber F in $\mathcal{M}_{g,2}$ of the point of $\mathcal{M}_g$ corresponding to X_g can be identified with the set of unordered pairs (x, y) of distinct points of X_g, modulo $\mathrm{Aut}(X_g)$. The intersection F_σ of this fiber with $\mathrm{Im}(\partial\mathcal{H})$ consists of those (x, y) such that there is a cover $f: X_g \to \mathbb{P}^1$ of type σ with $f(x) = f(y)$ and $f(x)$ not a branch point of f. Now assume X_g corresponds to a generic point of $\mathrm{Im}(\Phi_\sigma)$. Then by Lemma 2.3 and the hypothesis of infinite moduli degree, F_σ is Zariski dense in F. Since F_σ is the general fiber of the surjective map $\mathrm{Im}(\partial\mathcal{H}) \to \Phi_\sigma(\mathcal{H}_\sigma)$, it follows that $\dim \mathrm{Im}(\partial\mathcal{H}) = \dim F + \dim \Phi_\sigma(\mathcal{H}_\sigma) = \dim F + \textbf{mod-dim}(\sigma)$. This completes the proof. ∎

Consider the natural action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$ (by fractional linear transformations). It induces an action on $\mathcal{H}_\sigma$, with $\lambda \in \mathrm{PGL}_2(\mathbb{C})$ mapping $([f], (p_1, \ldots, p_r))$ to $([\lambda \circ f], (\lambda(p_1), \ldots, \lambda(p_r)))$. The closed subspace of $\mathcal{H}_\sigma$ defined by the conditions $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$ maps bijectively to the quotient $\mathcal{H}_\sigma / \mathrm{PGL}_2(\mathbb{C})$. Hence this quotient carries a natural structure of quasi-projective variety, and the map $\Phi_\sigma: \mathcal{H}_\sigma \to \mathcal{M}_g$ induces a morphism $\mathcal{H}_\sigma / \mathrm{PGL}_2(\mathbb{C}) \to \mathcal{M}_g$. (Clearly Φ_σ is constant on $\mathrm{PGL}_2(\mathbb{C})$-orbits.)

The dimension of (each component of) $\mathcal{H}_\sigma / \mathrm{PGL}_2(\mathbb{C})$ is $r - 3$. Thus if Φ_σ is dominant then $r - 3 \geq \dim \mathcal{M}_g$. This proves Remark 2.2(a). If $r - 3 > \dim \mathcal{M}_g$ then the general fiber of the map $\mathcal{H}_\sigma / \mathrm{PGL}_2(\mathbb{C}) \to \mathcal{M}_g$ is infinite. This proves Remark 2.2(b) (since two covers $f_1, f_2: X \to \mathbb{P}^1$ correspond to the same subfield of the function field of X if and only if f_1 is the composition of f_2 with an element of $\mathrm{PGL}_2(\mathbb{C})$).

For clarification, we now briefly discuss the general concept of moduli degree. This will not be needed elsewhere in the paper. The map $\mathcal{H}_\sigma / \mathrm{PGL}_2(\mathbb{C}) \to \mathcal{M}_g$ factorizes further over the action of S_r permuting the branch points (i.e., one can drop the ordering of the branch points. Actually, the version of the Hurwitz space without ordering of the branch points is more natural, see [V], Ch. 10, but for the purpose of this paper we need the ordering). Anyway, the natural definition of the moduli degree of σ is as follows: The degree of the induced map from the (irreducible) variety $\mathcal{H}_\sigma / (\mathrm{PGL}_2(\mathbb{C}) \times S_r)$ to $\mathcal{M}_g$. Thus the moduli degree of σ is the number of covers $f: X \to \mathbb{P}^1$ of type σ modulo $\mathrm{PGL}_2(\mathbb{C})$, where X corresponds to a (fixed) general point in the image of Φ_σ.

3. Covers with monodromy group A_n

We consider admissible tuples $\sigma = (\sigma_1, \ldots, \sigma_r)$ in S_n such that each σ_i is a double transposition (resp., 3-cycle). Then $r = n + g - 1 \geq n - 1$, where $g := g_\sigma$ (by Riemann–Hurwitz). Let $\mathrm{DT}(n, g)$ (resp., $\mathrm{TC}(n, g)$) be the set of these tuples σ; and let DTA(n, g) (resp., TCA(n, g)) be the subset consisting of those σ that generate A_n (the alternating group).

LEMMA 3.1: (i) *For each* $n \geq 4$ *(resp.,* $n \geq 6$*) the set* DT$(n, 0)$ *(resp.,* DTA$(n, 0)$*) is non-empty.*

(ii) *The set* $TCA(n, 0)$ *is non-empty for each* $n \geq 3$.

Proof: (i) For $n = 4$ take σ to consist of all double transpositions in A_4. For $n = 5$ take $\sigma = (\sigma_1, \ldots, \sigma_4)$ such that $\sigma_1\sigma_2$ $(= (\sigma_3\sigma_4)^{-1})$ is a 5-cycle. For $n = 6$ use GAP (or check otherwise).

Assume now σ is in DTA$(n, 0)$, and $n \geq 6$. We may assume $\sigma_r = (1, 2)(3, 4)$. Replacing σ_r by the two elements $(1, 2)(n, n + 1)$ and $(3, 4)(n, n + 1)$ yields a tuple in DTA$(n + 1, 0)$. This proves (i).

(ii) Here is a quick direct proof. For $n = 3$ take $\sigma = ((1, 2, 3), (1, 2, 3)^{-1})$. Assume now σ is in TCA$(n, 0)$, $n \geq 3$. We may assume $\sigma_1 = (1, 2, 3)$. Replacing σ_1 by the two elements $(n+1, 3, 1)$ and $(3, n+1, 2)$ yields a tuple in TCA$(n+1, 0)$.

LEMMA 3.2: *Both* DTA(n, g) *and* TCA(n, g) *contain a tuple of full moduli dimension if one of the following holds:*

(i) $g = 1$ *and* $n \geq 5$.

(ii) $g = 2$ *and* $n \geq 6$.

(iii) $g > 2$ *and* $n \geq 2g + 1$.

Proof: (i) See [FKK] for a proof of the TCA$(n, 1)$ case that does not use the stable compactification. For the DTA$(n, 1)$ case, we use induction on n.

We anchor our induction at $n = 5$. We choose $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$, where $\sigma_1 = \sigma_2 = (1, 2)(3, 4)$, $\sigma_3 = (1, 2)(4, 5)$, $\sigma_4 = (1, 4)(2, 5)$, and $\sigma_5 = (1, 5)(2, 4)$. If we coalesce the last two entries of σ we obtain $(\sigma_1, \sigma_1, \sigma_3, \sigma_3)$, which has orbits $\{1, 2\}$ and $\{3, 4, 5\}$. For a cover $X_1 \to \mathbb{P}^1$ of type σ, this means the following: When coalescing the last two branch points, X_1 degenerates into a nodal curve $\bar{X}$ with two components linked at one point P. Both components are non-singular curves of genus 1 (resp., 0). They both cover $\mathbb{P}^1$ with four branch points and of degree 2 (resp. 3). The point P ramifies in both covers. The nodal curve $\bar{X}$ is stably equivalent to its genus 1 component, and the latter constitutes the image in $\mathcal{M}_1$ of the cover $\bar{X} \to \mathbb{P}^1$ (as in the proof of Lemma 2.4). Clearly, every element of $\mathcal{M}_1$ can be obtained in this fashion. Thus the map $\mathcal{H}_\sigma \to \mathcal{M}_1$ is dominant because the boundary of $\mathcal{H}_\sigma$ already maps surjectively to $\mathcal{M}_1$.

Now assume $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a tuple in DTA$(n, 1)$, $n \geq 5$, of full moduli dimension. Write $\sigma_n = st$ where s, t are double transpositions in $S_{n+1} \setminus S_n$. Let

$\sigma' = (\sigma_1, \ldots, \sigma_{n-1}, s, t)$, a tuple in $\mathrm{DTA}(n+1, 1)$. Moreover, σ' has full moduli dimension because $\Phi_{\sigma'}$ restricted to the boundary component of $\mathcal{H}_{\sigma'}$ isomorphic to $\mathcal{H}_\sigma$ already maps dominantly to $\mathcal{M}_1$.

(ii) Same for both cases. So we only do the DT case. By (i), there is a tuple in $\mathrm{DTA}(n-1, 1)$ of full moduli dimension. Its length equals $n-1$, and $(n-1)-3 > 1 = \dim \mathcal{M}_1$; thus the tuple has infinite moduli degree by Remark 2.2(b). Then Lemma 2.4 produces a tuple in $\mathrm{DTA}(n, 2)$ of full moduli dimension.

(iii) Same for both cases. So we only do the DT case. First we settle the case $g = 3$, $n \geq 7$. By (ii), there is a tuple in $\mathrm{DTA}(n-1, 2)$ and of moduli dimension 3. Its length is n, and $n - 3 > 3 = \dim \mathcal{M}_2$; the claim follows from Remark 2.2(b) and Lemma 2.4.

Now suppose $g > 3$, $n \geq 2g+1$. Then $n - 1 \geq 2(g-1) + 2$. By induction we may assume there is a tuple in $\mathrm{DTA}(n-1, g-1)$ and of full moduli dimension. Its length is $r := n + g - 3$, and $r - 3 > 3(g-1) - 3 = \dim \mathcal{M}_{g-1}$; the claim follows again from Remark 2.2(b) and Lemma 2.4.

THEOREM 3.3: (i) *Let $g \geq 3$. Then each general curve of genus g admits a cover to $\mathbb{P}^1$ of degree n with monodromy group A_n such that all inertia groups are generated by double transpositions if and only if $n \geq 2g+1$.*

(ii) *For $n \geq 6$ (resp., $n \geq 5$), each general curve of genus 2 (resp., 1) admits a cover to $\mathbb{P}^1$ of degree n with monodromy group A_n such that all inertia groups are generated by double transpositions.*

(iii) *Assertions (i) and (ii) also hold for 3-cycles instead of double transpositions.*

Proof: In view of Lemma 3.2, it only remains to show that the condition $n \geq 2g+1$ in (i) is necessary. Indeed, if the general curve of genus g admits such a cover then an associated tuple of branch cycles is in $\mathrm{DTA}(n, g)$ and of full moduli dimension. Thus the claim follows from the necessary condition $r \geq 3g$ (Remark 2.2) since $r = n + g - 1$. The proof of (iii) is the same.

COROLLARY 3.4: *Let C be a general curve of genus $g \geq 4$. Then the monodromy groups of primitive covers $C \to \mathbb{P}^1$ are among the symmetric and alternating groups, and up to finitely many, all of these groups occur.*

Here a cover is called primitive if it does not factor non-trivially. The first assertion in the Corollary follows from [GM]. For the second assertion, the case of alternating groups follows from the Theorem, and the case of symmetric groups follows from Brill–Noether theory. See the following Remark for further details on the case of symmetric groups.

Remark 3.5 (Existence of covers with monodromy group S_n)*:* Assume $g \geq 2$. There exists an admissible tuple in S_n of genus g and full moduli dimension if and only if $2(n-1) \geq g$ (see the Introduction). It is stated in [Fr2], Principle 1.6, that if there exists such a tuple, then there is one that generates S_n (and even consists only of transpositions — the case of simple covers). We couldn't find a reference to a proof of that, so we supply the argument here.

Note first that if we append two copies of the transposition $(n, n+1)$ to an admissible tuple σ in S_n, we obtain an admissible tuple σ' in S_{n+1} of the same genus. It follows by similar arguments as above (e.g., in the proof of Lemma 3.2(i)) that if σ has full moduli dimension, so has σ'. This reduces the question to the case of minimal n, i.e., $2(n-1) = g$ if g is even and $2(n-1) = g+1$ if g is odd.

Suppose $\sigma = (\sigma_1, \ldots, \sigma_r)$ in S_n has full moduli dimension, and let $g = g_\sigma$. Firstly assume $2(n-1) = g$. Then by Riemann–Hurwitz and Remark 2.2 we get $3g = 2(n+g-1) = \sum_{i=1}^{r} \mathrm{Ind}(\sigma_i) \geq r \geq 3g$. Thus each σ_i has index 1, hence is a transposition. Thus $G = \langle \sigma_1, \ldots, \sigma_r \rangle$ is a transitive subgroup of S_n generated by transpositions, i.e., $G = S_n$.

If $2(n-1) = g+1$ then we get that all but possibly one of the σ_i's are transpositions. Since those transpositions generate G, we get again $G = S_n$.

Question: In the minimal case $2(n-1) = g$, the moduli degree of the corresponding tuple σ of transpositions is finite, i.e., the general curve of genus g admits only finitely many covers of type σ. This finite number has been computed by Castelnuovo, see [ACGH], p. 211. What are the analogous numbers in the minimal case $n = 2g+1$ for 3-cycle tuples resp., double transposition tuples in alternating groups?

4. Braid orbits of admissible tuples

The **braid orbit** of a tuple σ in S_n is the smallest set of tuples in S_n that contains σ and is closed under (component-wise) conjugation and under the braid operations

$$(g_1, \ldots, g_r)^{Q_i} = (g_1, \ldots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots, g_r)$$

for $i = 1, \ldots, r-1$.

Let σ, σ' be admissible tuples in S_n of length r. Let $f\colon X \to \mathbb{P}^1$ be a cover of type σ. Then f is of type σ' if and only if σ' lies in the braid orbit of σ. In other words, for the associated Hurwitz spaces we have $\mathcal{H}_\sigma = \mathcal{H}_{\sigma'}$ if and only if

σ' lies in the braid orbit of σ (see [FrV], [V], Ch. 10). Thus the above notions of moduli dimension, moduli degree etc. depend only on the braid orbit of σ. So from now on we will speak of the moduli dimension of a braid orbit, etc.

4.1 BRAID ORBITS OF 2-CYCLE TUPLES. Admissible tuples in S_n of fixed length that consist only of transpositions form a single braid orbit (by Clebsch 1872, see [V], Lemma 10.15). They correspond to the so-called **simple covers**. Their braid orbit has full moduli dimension if and only if $2(n-1) \geq g$, where $g = g_\sigma$ (see the remarks in the Introduction).

4.2 BRAID ORBITS OF 3-CYCLE TUPLES. Now consider tuples that consist only of 3-cycles. Recall our notation $\mathrm{TC}(n,g)$ for the set of those (admissible) tuples with fixed parameters n, g. Assume $n \geq 5$. Note that $\mathrm{TC}(n,g) = \mathrm{TCA}(n,g)$ (i.e., each such tuple generates A_n) by [Hup], Satz 4.5.c and the fact that a transitive group generated by 3-cycles must be primitive. The corresponding covers have been studied by Fried [Fr1]. Serre [Se1], [Se2] considered certain generalizations. Fried proved that $\mathrm{TC}(n,g)$ (is non-empty and) consists of exactly two braid orbits (resp., one braid orbit) if $g > 0$ (resp., $g = 0$). Let

$$\{\pm 1\} \to \hat{A}_n \to A_n$$

be the unique non-split degree 2 extension of A_n. Each 3-cycle $t \in A_n$ has a unique lift $\hat{t} \in \hat{A}_n$ of order 3. For $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathrm{TC}(n,g)$ we have $\hat{\sigma}_1 \cdots \hat{\sigma}_r = \pm 1$. The value of this product is called the **lifting invariant** of σ. It depends only on the braid orbit of σ. For $g = 0$ the lifting invariant is $+1$ if and only if n is odd (by [Fr1] and [Se1]). For $g > 0$ the two braid orbits on $\mathrm{TC}(n,g)$ have distinct lifting invariant.

Now we can refine Theorem 3.3 as follows.

THEOREM 4.1: *Assume $n \geq 6$, $g > 0$ and $n \geq 2g+1$. Then both braid orbits on $\mathrm{TC}(n,g)$ have full moduli dimension.*

Proof: The claim holds for $g = 1$ by [FKK], Comment 0. Now suppose in the situation of Lemma 2.4, $\tilde{\sigma}$ is a tuple in A_n with $\sigma_{r+1} = \sigma_{r+2}^{-1} = (n-1, n, n+1)$. Then clearly σ and $\tilde{\sigma}$ have the same lifting invariant. Thus the proof of Lemma 3.2 also shows the present refinement, since it iterates the construction of Lemma 2.4.

5. The exceptional cases in genus 3

Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be an admissible tuple in S_n, and $g := g_\sigma \geq 3$. Assume σ satisfies the necessary condition $r \geq 3g$ for full moduli dimension. Assume further σ generates a primitive subgroup G of S_n. If $g \geq 4$ then $G = S_n$ or $G = A_n$ by [GM] and [GS]. If $g = 3$ and G is not S_n or A_n then one of the following holds (see [GM], Theorem 2):

(1) $n = 7$, $G \cong GL_3(2)$.

(2) $n = 8$, $G \cong AGL_3(2)$ (the affine group).

(3) $n = 16$, $G \cong AGL_4(2)$.

Recall that $GL_3(2)$ is a simple group of order 168. It acts doubly transitively on the 7 non-zero elements of $(\mathbb{F}_2)^3$. The affine group $AGL_m(2)$ is the semi-direct product of $GL_m(2)$ with the group of translations; it acts triply transitively on the affine space $(\mathbb{F}_2)^m$.

In cases (1) and (3), the tuple σ consists of 9 transvections of the respective linear or affine group. (A transvection fixes a hyperplane of the underlying linear or affine space point-wise.) In case (2), either σ consists of 10 transvections or it consists of 8 transvections plus an element of order 2, 3 or 4 (where the element of order 2 is a translation).

Remark 5.1: The tuples in case (1) form a single braid orbit on DT(7,3). This braid orbit has full moduli dimension by the Theorem below.

Proof: We show that tuples of 9 involutions generating $G = GL_3(2)$ (with product 1) form a single braid orbit. This uses the BRAID program [MSV]. Direct application of the program is not possible because the number of tuples is too large.

We first note that if 9 involutions generate G, then there are 6 among them that generate already (since the maximal length of a chain of subgroups of G is 6). We can move these 6 into the first 6 positions of the tuple by a sequence of braids. Now we apply the BRAID program to 6-tuples of involutions generating G (but not necessarily with product 1). We find that such tuples with any prescribed value of their product form a single braid orbit. By inspection of these braid orbits, we find that each contains a tuple whose first two involutions are equal, and the remaining still generate G. This reduces the original problem to showing that tuples of 7 involutions with product 1, generating G, form a single braid orbit. The BRAID program did that. ∎

In cases (1) and (2), the transvections yield double transpositions in S_n. Thus again Lemma 2.4 can be used to show there actually exist such tuples that have full moduli dimension. Case (3) requires a more complicated argument which will be worked out later.

THEOREM 5.2: *Each general curve of genus 3 admits a cover to $\mathbb{P}^1$ of degree 7 (resp., 8) and monodromy group $GL_3(2)$ (resp., $AGL_3(2)$), branched at 9 (resp., 10) points of $\mathbb{P}^1$, such that all inertia groups are generated by double transpositions.*

Proof: Let G be a (doubly) transitive subgroup of S_7 isomorphic to $GL_3(2)$. Let H ($\cong S_4$) be a point stabilizer in G. View H as a subgroup of S_6 via its (transitive) action on the other 6 points. In 5.1 below, we show there is a tuple τ in DT(6, 2) of full moduli dimension that generates this subgroup H of S_6. This tuple has length 7, hence has infinite moduli degree by Remark 2.2(b). Choose a double transposition in G that is not in H, and append two copies of it to the tuple τ. By Lemma 2.4, this yields a tuple $\sigma \in$ DT(7, 3) of full moduli dimension, satisfying (1).

The group $GL_3(2)$ is the stabilizer of 0 in the transitive action of $AGL_3(2)$ on the 8 points of $(\mathbb{F}_2)^3$. Replacing the last entry σ_9 of the above tuple σ by two double transpositions from $AGL_3(2)$ that are not in $GL_3(2)$ and have product σ_9, yields a tuple in DT(8, 3) satisfying (2). This tuple has full moduli dimension because already the boundary of the corresponding Hurwitz space maps dominantly to $\mathcal{M}_3$.

5.1 CERTAIN COVERS OF DEGREE 6 FROM THE GENERAL CURVE OF GENUS 2 TO $\mathbb{P}^1$. Let τ_1, τ_2, τ_3 be the three double transpositions in $H := S_4$. Let ρ_1 and ρ_2 be transpositions in H generating an S_3-subgroup. Then the tuple

$$\tau = (\tau_1, \tau_2, \tau_3, \rho_1, \rho_1, \rho_2, \rho_2)$$

generates H. View H as a subgroup of S_6 as in the proof of Theorem 5.2. Then τ becomes an element of DT(6, 2) (since all involutions of $GL_3(2)$ act as double transpositions on the 7 points).

Now consider a cover $f\colon X \to \mathbb{P}^1$ of type τ. Note that H is an imprimitive subgroup of S_6, permuting 3 blocks of size 2. The kernel of the action of H on these 3 blocks equals $\{1, \tau_1, \tau_2, \tau_3\}$. Thus f factors as $f = hg$ where $g\colon X \to \mathbb{P}^1$ is of degree 2 (the hyperelliptic map on the genus 2 curve X) and $h\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a simple cover of degree 3 (i.e., its tuple of branch cycles consists of 4 involutions in S_3). Let $p_i \in \mathbb{P}^1$, $i = 1, 2, 3$ be the branch point of f corresponding to τ_i. Then p_i has 3 distinct pre-images x_i, y_i, z_i under h. We may assume $p_1 = 0 = x_3$,

$p_2 = \infty = y_3$, $p_3 = 1 = x_1$. Then h is of the form

$$h(x) = \frac{(x-1)(x-y_1)(x-z_1)}{(x-x_2)(x-y_2)(x-z_2)}.$$

Exactly one of x_i, y_i, z_i, say z_i, is unramified under g. Thus $x_1 = 1, y_1, x_2, y_2, x_3 = 0, y_3 = \infty$ are the 6 branch points of the hyperelliptic map g. It is well-known that (the PGL 2-orbit of) this 6-set determines the isomorphism class of the genus 2 curve X. Now we are ready to prove:

LEMMA 5.3: *The tuple τ has full moduli dimension.*

Proof: It suffices to show that for each choice of y_1', x_2', y_2' sufficiently close to y_1, x_2, y_2, respectively (in the complex topology), the following holds: There are z_1', z_2' close to z_1, z_2, respectively, such that the map

$$h'(x) = \frac{(x-1)(x-y_1')(x-z_1')}{(x-x_2')(x-y_2')(x-z_2')}$$

composed with the double cover $g'\colon X' \to \mathbb{P}^1$ branched at $y_1', x_2', y_2', 0, \infty, 1$ is a cover of type τ. This follows by continuity once we know that the condition $h'(0) = 1$ $(= h'(\infty))$ is preserved. But this condition $h'(0) = 1$ is easy to achieve: We can view it as defining z_2' (after free choice of z_1').

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